MATH 4010: Functional Analysis: Complementary Exercise
December 2019

1. Let $-\infty < a < b < \infty$ and let $(\mu_k)_{k=0}^{\infty}$ be a sequence of real numbers. Show that the following statements are equivalent.

   (i) There is a bounded variation $g : [a, b] \to \mathbb{R}$ such that
   $$\int_a^b x^k dg(x) = \mu_k$$
   for all $k = 0, 1, 2, ...$

   (ii) There is $C > 0$ such that
   $$|\sum_{k=0}^{N} a_k \mu_k| \leq C \max\{|\sum_{k=0}^{N} a_k x^k| : x \in [a, b]\}$$
   for all $a_k \in \mathbb{R}$ and for all $N = 0, 1, 2, ...$

2. Let H be a Hilbert space. Let $u : H \to H$ be a bounded linear operator. Show that the followings are equivalent

   (i) u^*u is a projection.

   (ii) uu^* is a projection.

   (iii) There are closed subspaces E and F of H such that the restriction of u on E is an isometry from E onto F with $\ker u = E^{\perp}$.

3. Let X be a normed space and let S_{X^*} be the closed unit sphere of X^*. Suppose that there is $0 < r < 1$ such that $S_{X^*} \subseteq \bigcup_{k=1}^{n} B(x_k^*, r)$ for some $x_1^*, ..., x_n^*$ in X^* with $\|x_k^*\| = 1$ for all $k = 1, ..., n$.
   Define a linear map $T : X \to c_0$ by
   $$T(x) = (x_1^*(x), ..., x_n^*(x), 0, ...) \in c_0.$$

   (i) Show that $\|T\| = 1$

   (ii) Show that $\|x\| \leq \frac{1}{1-r}\|Tx\|$ for all $x \in X$.

   (iii) Show that the operator is an isomorphism from X onto a subspace $T(X)$ of c_0 and $\|T^{-1}\| \leq 1/(1-r)$.

   (iv) If Y is a Banach space, show that for any $\varepsilon > 0$ and any finite dimensional subspace E of Y, there is an isomorphism T from E onto a subspace F of c_0 such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

End